Approximation theorem for the self-focusing Nonlinear Schrödinger Equation and for the periodic curves in $\mathbf{R}^3$.

P.G.Grinevich [1]

Landau Institute for Theoretical Physics, Kosygina 2, Moscow, 117940, Russia.
e-mail: pgg@landau.ac.ru

Dedicated to V.E.Zakharov's 60th birthday

Abstract

It is shown, that any sufficiently smooth periodic solution of the self-focusing Nonlinear Schrödinger equation can be approximated by finite-gap ones with an arbitrary small error. As a corollary an analogous result for the motion of closed curves in $\mathbb{R}^3$ guided by the Filament equation is proved. This equation describes the dynamics of very thin filament vortices in a fluid.

One of the basic questions of the finite-gap theory is the following: are the finite-gap solutions of a given equation sufficiently generic or they belong to some special subclass? To answer this question it is reasonable to check if arbitrary periodic in spatial variables solution can be approximated by finite-gap ones.

The study of simplest examples shows, that the uniform approximation for all x and t is impossible, because to do it we have to keep all space and time frequencies simultaneously, and we have too many constraints on the deformations to fulfill all of them.

Therefore it is natural to ask the following questions:

1) Let us have a smooth periodic in spatial variables solution and an arbitrary compact domain U in the (x,t) space. Is it possible to approximate our solution on U by finite-gap ones with arbitrary small error? The approximating

[1] This work was supported by the INTAS grant No. 96-770 and by the Russian Foundation for Basic Research grant No. 98-01-01161.

solutions are allowed to be non-periodic. We shall call such approximations **local**.

2) Let us have a smooth periodic in spatial variables solution and an arbitrary compact domain U in the t space. Is it possible to approximate our solution by periodic in x finite-gap ones with the same x-periods for all x and $t \in U$? We shall call such approximations **periodic**.

Of course any periodic approximation is automatically local, but the characterization of periodic finite-gap solutions is usually sufficiently complicated, therefore the transition from local approximations to periodic ones may be rather nontrivial.

In our text we construct **periodic** approximations for the self-focusing Nonlinear Schrödinger equation (SfNLS)

$$iq_t + q_{xx} + 2q^2\bar{q} = 0, \quad \text{where} \tag{1}$$

$q = q(x,t)$ is a complex-valued function of two real variables, and for the Filament equation

$$\frac{\partial \vec{\gamma}(s,t)}{\partial t} = k(x,t)\vec{b}(s,t), \quad \text{where} \tag{2}$$

$\vec{\gamma}(s,t)$ is a t-dependent family of smooth curves in $\mathbb{R}^3$, s is the natural parameter (i.e. $|\partial_s \vec{\gamma}(s,t)| \equiv 1$), $\vec{b}(s,t)$, $k(s,t)$ denote the binormal vector from the Frenet reper and the curvature function respectively.

The first periodic approximation theorem was proved in 1975 by Marchenko and Ostrovskii [9] for the real Korteveg de Vries (KdV) equation. The method of [9] is based on the theory of conformal maps and it can be naturally extended to some soliton systems including the defocusing NLS. But for other systems the question is still open and the answer depends on the equation. For example, from results of Krichever [8] it is rather clear that any periodic Kadomthsev-Petviashvili II (KP II) solution allows finite-gap approximations, but it is likely that for KP I it is not so.

Direct attempts to generalize the approach of [9] to SfNLS meet the following problems

(1) The space of spectral curves corresponding to real periodic g-gap KdV solutions is topologically $\mathbb{R} \oplus (\mathbb{R}^+)^g$. But in the SfNLS case this space it the real part of some ramified covering of $\mathbb{C}^{g+1}$, and the structure of the Marchenko-Ostrovskii conformal map is essentially more complicated. To avoid a detailed study of the parameters space we use the method of

isoperiodic deformations suggested by M.Schmidt and author in [3].

(2) In the KdV case the characterization of admissible divisors is very simple: the n-th point of divisor is located at an arbitrary point of the n-th compact real oval. In the SfNLS case the characterization is less explicit (see below) and we have to describe how we vary the divisor after perturbing the spectral curve to preserve the admissibility.

(3) Dubrovin equations for the real KdV are non-singular therefore a small change of parameters and starting point slightly affects the solution. But in the SfNLS case Dubrovin equations may have singularities, and the solutions may have branch points (nevertheless the corresponding SfNLS potential is smooth). Therefore we have to check that our variations of spectral data do not change the solution too much. To do it we introduce some new "symmetric" variables.

Let us recall some basic facts from the SfNLS theory. The scattering transform for NLS was found in 1971 by Zakharov and Shabat [15]. Finite-gap NLS solutions were first constructed in 1976 by Its and Kotljarov [6]. The characterization of SfNLS admissible divisors as well as a proof that all solutions with reduction (6) are automatically nonsingular were obtained by Cherednik [1]. Infinite-gap periodic problem for matrix operators including the NLS L-operator was studied by M.Schmidt [12]. A lot of useful information about the NLS theory including the Hamiltonian theory is contained in the book [2] by Faddeev and Takhtadjan. Finite-gap NLS theory is discussed in details in the article [11] by Previato. Effictivisation of low genus formulas by NLS was studied by Kamchatnov [7].

The zero-curvature representation for SfNLS reads as:

$$\frac{\partial \Psi(\lambda,x,t)}{\partial x} = U(\lambda,x,t)\Psi(\lambda,x,t), \quad \frac{\partial \Psi(\lambda,x,t)}{\partial t} = V(\lambda,x,t)\Psi(\lambda,x,t), \tag{3}$$

where $\Psi(\lambda,x,t)$ is a 2-component vector,

$$\Psi(\lambda,x,t) = \begin{bmatrix} \psi_1(\lambda,x,t) \\ \psi_2(\lambda,x,t) \end{bmatrix}, \tag{4}$$

$U(\lambda,x,t)$, $V(\lambda,x,t)$ are the following 2×2 matrices:

$$U(\lambda,x,t) = \begin{bmatrix} i\lambda & iq(x,t) \\ ir(x,t) & -i\lambda \end{bmatrix}, V(\lambda,x,t) = -2\lambda U(\lambda,x,t) + \begin{bmatrix} iqr & -q_x \\ r_x & -iqr \end{bmatrix}, \tag{5}$$

$$r(x,t) = \bar{q}(x,t). \tag{6}$$

We shall assume that $q(x,t)$ is periodic in x with the period 1

$$q(x+1,t) \equiv q(x,t). \tag{7}$$

We shall fix our attention on the spectral transform for a fixed $t = t_0$, therefore starting from this moment we shall omit t in our notations.

The Bloch eigenfunction $\Psi(\lambda, x)$ is by definition the common eigenfunction of $L = \partial_x - U(\lambda, x)$ and the shift operator

$$L\Psi(\lambda, x) = 0, \quad \Psi(\lambda, x+1) = e^{ip(\lambda)}\Psi(\lambda, x). \tag{8}$$

Equation (8) defines $\Psi(\lambda, x)$ up to a constant factor. We fix it assuming

$$\Phi(\lambda, 0) \equiv 1, \quad \text{where} \quad \Phi(\lambda, x) = \psi_1(\lambda, x) + \psi_2(\lambda, x). \tag{9}$$

The function $p(\lambda)$ is defined up to adding $2\pi n$, $n \in \mathbb{Z}$. It is called the quasi-momentum.

To calculate the Bloch function we have to diagonalize the 2×2 monodromy matrix $T(\lambda)$, which is an entire function of λ. The eigenfunctions of $T(\lambda)$ lie on a two-sheeted covering Γ of the λ-plane. Γ is called spectral curve. Denote the permutation of sheets of Γ by σ. $\det T(\lambda) \equiv 1$, therefore $p(\gamma) + p(\sigma\gamma) \equiv 0 (\mathrm{mod} 2\pi)$. (We shall denote points of Γ by γ and the projection $\Gamma \to \mathbb{C}$ by $\mathcal{P}$, $\lambda = \mathcal{P}\gamma$). $p(\gamma)$ is a locally holomorphic multivalued function on Γ, $dp = (dp(\lambda)/d\lambda)d\lambda$ is a holomorphic differential on the finite part of Γ, $\sigma(dp) = -dp$. $p(\gamma) \sim \pm\lambda$ as $\lambda \to \infty$ (as an asymptotic series), therefore Γ is compactified by 2 infinite points ∞_+, ∞_-, $\sigma\infty_+ = \infty_-$, $\sigma\infty_- = \infty_+$, $p(\gamma) \sim \pm\lambda$ as $\gamma \to \infty_\pm$ respectively.

A point $\lambda \in \mathbb{C}$ is called **regular** if $p(\gamma) \neq 0 (\mathrm{mod} \pi)$, where $\mathcal{P}\gamma = \lambda$ and **irregular** otherwise. Let λ_k be an irregular point. The Tailor expansion of $\mathrm{tr}\, T(\lambda)$ reads as: $\mathrm{tr}\, T(\lambda) = \pm 2 + T_k^n(\lambda - \lambda_k)^n + \ldots$. Let us call n the order of the point λ_k, $n = \mathrm{ord}_q(\lambda_k)$. ($q$ means that the order is defined in terms of the quasimomentum function). λ_k is a **branch point** of Γ if n is odd and a **double point** of Γ if n is even. A branch point is called **simple** if $n = 1$. If the opposite is not stated explicitly we have **one** point of Γ over each double point. To have an uniform representation for our equations we shall treat an irregular point of order n as the result of fusion n simple branch point.

If $\Psi(\lambda, x)$ is a Bloch solution of (8), then

$$\Psi^+(\bar\lambda, x) = \begin{bmatrix} \bar\psi_2(\lambda, x) \\ -\bar\psi_1(\lambda, x) \end{bmatrix}, \tag{10}$$

is also a Bloch solution of (8) with the quasimomentum $p^+(\bar\lambda) = -\bar p(\lambda)$. Therefore Γ has the following antiholomorphic involutions $\gamma \to \sigma\bar\gamma$ and $\gamma \to \bar\gamma$. (We assume that $p(\bar\gamma) = \bar p(\gamma)$, $p(\sigma\bar\gamma) = -\bar p(\gamma)$). For real λ $T(\lambda)$ is an unitary matrix, $p(\lambda) \in \mathbb{R}$, and Γ has no real branch points (but may have real double points).

Let $\{E_k\}$, $\{E_k^+\}$ be the lists of all irregular points, the index k takes all integer values. We assume that

(1) $\operatorname{Im} E_k \geq 0$.
(2) $E_k^+ = \overline{E_k}$.
(3) If $\operatorname{ord}_q(\lambda_k) = n$ the point λ_k has exactly n entries in our lists. For example if $\operatorname{ord}_q(\lambda_k) = 4$ and $\lambda_k \in \mathbb{R}$ then we have exactly 2 integers k_1, k_2 such that $E_{k_1} = E_{k_2} = E_{k_1}^+ = E_{k_2}^+$.

Lemma 1 *It is possible to enumerate the irregular points so, that for sufficiently large $|k|$*

(1) $E_k = (\pi \operatorname{sgn} k) \cdot \sqrt{k^2 - I_1(q)} + o\left(\frac{1}{k}\right)$ *where* $I_1(q) = \int_0^1 q(x)\bar q(x)dx$.
(2) $E_k - E_k^+ \to 0$ *faster than any degree of* k^{-1} *as* $|k| \to \infty$ *(We assume* $q(x)$ *to be smooth).*

The next important object for us is the set of **zeroes of the quasimomentum differential** dp. They are invariant under the involution σ therefore we shall consider their projections to the λ-plane instead. Denote them by α_k where k takes all integer values. As above we use the following agreement

(1) If λ_k is a regular points it has n entries in the list $\{\alpha_k\}$ where n is the order of zero of dp at one sheet.
(2) If λ_k is an irregular points it has $\left[\frac{\operatorname{ord}_q(\lambda_k)}{2}\right]$ entries to the list $\{\alpha_k\}$ where $[\,]$ denotes the integer part.

Lemma 2 *It is possible to enumerate the points α_k so, that for sufficiently large $|k|$*

(1) $\operatorname{Im} \alpha_k = 0$.
(2) $\alpha_k = \operatorname{Re} E_k + o(\operatorname{Im} E_k)$.

Let us define now the "second part" of the spectral data – the **divisor of poles** of the Bloch function.

Let $\tilde\Psi(\gamma, x)$ denote a Bloch eigenfunction of L with some non-singular locally holomorphic normalisation (of course $\Psi(\gamma, x) = \tilde\Psi(\gamma, x)/\tilde\Phi(\gamma, 0)$ where $\tilde\Phi(\gamma, x) = \tilde\psi_1(\gamma, x) + \tilde\psi_2(\gamma, x)$. Consider the Wronskian of the Bloch functions $\tilde W(\gamma) = \tilde\psi_1(\gamma, x)\tilde\psi_2(\sigma\gamma, x) - \tilde\psi_2(\gamma, x)\tilde\psi_1(\sigma\gamma, x)$. It is defined up to a non-zero holomorphic multiplier and does not wanish at regular points. Let λ_k be an

irregular point. We have $W(\lambda) = \pm w_k^m (\lambda - \lambda_k)^{m/2}(1 + o(1))$ where m is even if λ_k is a double point and odd if λ_k is a branch point, $m \geq 0$. Denote m by $\mathrm{ord}_b(\lambda_k)$. It is easy to check that $\mathrm{ord}_b(\lambda_k) \leq \mathrm{ord}_q(\lambda_k)$. A double point λ_k is called **removable** if $\mathrm{ord}_b(\lambda_k) = 0$. It is well-known that removable double points can be treated as regular points and we can forget about them.

The divisor of Bloch function zeroes is a list of points of Γ $\{\gamma_k(x)\}$ where k takes all integer values such that each zero of $\tilde{\Phi}(\gamma, x)$ generates l entries to this list if l is the multiplicity of it and each irregular point generates $(\mathrm{ord}_q - \mathrm{ord}_b)/2$ entries. **The divisor of Bloch function poles** $\{\gamma_k\}$ coinsides with the divisor of Bloch function zeroes taken at the point $x = 0$.

Lemma 3 *(1) The spectral curve Γ has only finite number of non-removable double points and degenerate branch points.*
(2) All real double points are removable.

Lemma 4 *It is possible to enumerate the points γ_k so, that for sufficiently large $|k|$ $\mathcal{P}\gamma_k = \mathrm{Re}\, E_k + O(\mathrm{Im}\, E_k)$.*

It is well-known, that the spectral curve and the divisor of poles completely define the potential $q(x)$. To reconstruct the potential we can use **Dubrovin equations**

$$\frac{\partial}{\partial x}\lambda_j(x) = -2i\left[\lambda_j(x) + \sum_{k=-\infty}^{\infty}\left(\frac{E_k + E_k^+}{2} - \lambda_k(x)\right)\right]\nu_j(x), \tag{11}$$

where

$$\nu_j(x) = \sqrt{(\lambda_j(x) - E_j)(\lambda_j(x) - E_j^+)}\prod_{k\neq j}\frac{\sqrt{(\lambda_j(x) - E_k)(\lambda_j(x) - E_k^+)}}{\lambda_j(x) - \lambda_k(x)}, \tag{12}$$

and the reconstruction formula

$$q(x) = \sum_{k=-\infty}^{\infty}\left(\frac{E_k + E_k^+}{2} - \lambda_k(x)\right) + \sum_{j=-\infty}^{\infty}\nu_j(x). \tag{13}$$

The infinite sums and products in the formulas above perfectly converge.

Let us recall the characterization of divisors corresponding to operators with the reduction (6). Consider the following 1-form on Γ: $\Omega(\gamma, x) = \omega(\gamma, x)d\lambda$, where

$$\omega(\gamma, x) = \frac{(\tilde{\psi}_1(\gamma, x) + \tilde{\psi}_2(\gamma, x))(\tilde{\psi}_2(\sigma\gamma, x) - \tilde{\psi}_1(\sigma\gamma, x))}{\tilde{\psi}_1(\gamma, x)\tilde{\psi}_2(\sigma\gamma, x) - \tilde{\psi}_2(\gamma, x)\tilde{\psi}_1(\sigma\gamma, x)}. \tag{14}$$

Denote by $U(R)$ be the domain $|\lambda| < R$ in Γ. Consider the following function $\omega_R(\gamma, x) = \omega(\gamma, x) \prod_{k:|E_k|<R} \sqrt{(\lambda - E_k)(\lambda - E_k^+)}$. Denote by $D(\omega_R, x)$ the divisor of zeroes of $\omega_R(\gamma, x)$ and by $D(\omega, x)$ the limit of $D(\omega_R, x)$ as $R \to \infty$.

Lemma 5 *$D(\omega, x)$ coincide with the set $\{\gamma_k(x), \bar{\gamma}_k(x)\}$, where $\{\gamma_k(x)\}$ is the divisor of Bloch function zeroes.*

Let δ_j be an arbitratry collection of pairwise distinct real points such that for sufficiently large $|j|$ $\delta_j = \operatorname{Re} E_j$.

Lemma 6 *(1) The form Ω reads as $\Omega = [1 - \tilde{\kappa}(\gamma, x)]\, d\lambda$*

$$\tilde{\kappa}(\gamma, x) = \sum_{j=-\infty}^{\infty} \frac{\kappa_j(x)}{\sqrt{(\lambda - E_j)(\lambda - E_j^+)}} \prod_{k \neq j} \frac{\lambda - \delta_k}{\sqrt{(\lambda - E_k)(\lambda - E_k^+)}}, \quad (15)$$

where $\kappa_j(x)$ are some real functions of x.
(2) $|\kappa(\gamma, x)| \leq 1$ for all $x \in \mathbb{R}$, $\gamma \in \mathbb{R}$.

In particular $\kappa(\delta_j, x)| \leq 1$ for all j. It gives us the following estimate on the functions $\kappa_j(x)$:

$$|\kappa_j(x)| \leq \left| \sqrt{(\delta_j - E_j)(\delta_j - E_j^+)} \prod_{k \neq j} \frac{\sqrt{(\delta_j - E_k)(\delta_j - E_k^+)}}{\delta_j - \delta_k} \right| \leq |\delta_j - E_j|\, C_\Gamma, (16)$$

where C_Γ is a positive constant, depending only on the spectral curve. In particular, if we have a removable double point $E_k = E_k^+ = \delta_k$, then $\kappa_k(x) \equiv 0$.

Lemma 7 *Let $\kappa_k(0)$ be a collection of real numbers such, that $|\kappa(\gamma, 0)| \leq 1$ for all $\gamma \in \Gamma$, where $\kappa(\gamma, 0)$ is defined by (15), $D(\omega, 0)$ be the corresponding divisor, $\{\gamma_j(0)\}$ be any set of points such that $D(\omega, 0) = \{\gamma_k(0), \bar{\gamma}_k(0)\}$. Then the corresponding operator L-operator satisfy (6), and the potential $q(x)$ is nonsingular.*

For us the following definition will be convenient: potential $q(x)$ is called **finite-gap** if $E_j = E_j^+$ for all $|j| \geq J_0$. Then all points $E_j = E_j^+$ are removable double points, $\alpha_j = \gamma_j(x) = E_j$ for all $|j| \geq J_0$ and Γ has only finite number of branch points and non-removable double points. Finite-gap solutions of soliton equations were first introduced by Novikov in 1974 for KdV [10]. The corresponding solutions can be written explicitly in terms of Riemann θ-functions.

The first step of the approximation procedure is to construct a finite-gap deformation of Γ generating solutions with the same period. To do it we need the following lemma proved by M.Schmidt and the author in [3].

Lemma 8 *Let $\alpha_k \in \mathbb{R}$ be the projection of a zero of the quasimomentum such, that $\alpha_k \neq \alpha_j$ for $j \neq k$, $\alpha_k \neq E_j$, $\alpha_k \neq E_j^+$, for all j. Consider the following system of ODE's, associated with the point α_k:*

$$\frac{\partial E_j}{\partial \tau} = -\frac{c_k(\tau)}{E_j - \alpha_k}, \quad \frac{\partial E_j^+}{\partial \tau} = -\frac{c_k(\tau)}{E_j^+ - \alpha_k},$$

$$\frac{\partial \alpha_j}{\partial \tau} = -\frac{c_k(\tau)}{\alpha_j - \alpha_k} \quad \text{for} \quad j \neq k, \tag{17}$$

$$\frac{\partial \alpha_k}{\partial \tau} = c_k(\tau) \left[\sum_{j \neq k} \frac{1}{\alpha_j - \alpha_k} - \frac{1}{2} \sum_{j=-\infty}^{\infty} \left(\frac{1}{E_j - \alpha_k} + \frac{1}{E_j^+ - \alpha_k} \right) \right],$$

where $c_k(\tau)$ is an arbitrary real function of τ.

Denote by $\Gamma(\tau)$ the solution of (17) with the initial value $\Gamma(0) = \Gamma$, where the spectral curve Γ corresponds to a periodic with the period 1 potential $q(x)$ (of course this solution is defined only in some neighborhood of zero $U(0)$). Then for all $\tau \in U(0)$ the curve $\Gamma(\tau)$ generates periodic with the period 1 potentials (the x-quasifrequencies of the potentials do not depend on the divisor).

Let $|k|$ be sufficiently large. Then using this deformation we can merge the pair E_k, E_k^+ to a removable double point, and the corresponding shift of all points E_j, E_j^+, α_j with $j \neq k$ is of order $o(\operatorname{Im} E_k)$. Therefore applying this deformation to all k such that $|k| \geq K$, where $K > 0$ is a sufficiently large integer, we obtain a finite-gap spectral curve Γ_K (it is almost evident that the superposition of infinitely many deformations perfectly converges).

Lemma 9 *For any $\epsilon > 0$ there exists a K such, that*

(1) $|E_j - \tilde{E}_j| < \epsilon$, $|E_j^+ - \tilde{E}_j^+| < \epsilon$, $|\alpha_j - \tilde{\alpha}_j| < \epsilon$, for all j.
(2) $|\operatorname{Im}(E_j - \tilde{E}_j)| < \epsilon |\operatorname{Im} E_j|$, for all j such, that $|j| < K$.

where $\tilde{E}_j$, $\tilde{E}_j^+$, $\tilde{\alpha}_j$ are the branch points of the curve Γ_K and the quasimomentum zeroes respectively.

We have constructed a family of finite-gap curves Γ_K approximating the curve Γ. Let us discuss now the admissible divisors.

Lemma 10 *There exists a pair of positive integer constants K_1, K_2 such, that for all $K \geq K_2$ the points of any admissible divisor γ_k on Γ_K can be enumerated so, that*

(1) For all k such that $|k| \leq K_1$ $|\lambda_k| < K_1 + 1/10$.
(2) For all k such that $|k| > K_1$ $|(\lambda_k - \tilde{\delta}_k)| \leq \operatorname{Im} \tilde{E}_k$, and $|\tilde{\delta}_k| > K_1 + 1 - 1/10$.

The proof follows from the characterization of admissible divisors given by Lemmas 5-7.

Equations (11)-(12) have singularities at the right-hand side. To simplify the structure of Dubrovin equation it is convenient to introduce the following new variables:

(1) $s_k(x)$, $q_k(x)$, $1 \leq k \leq 2K_1+1$ – the first $2K_1+1$ expansion coefficients at ∞ of the function

$$\Xi(\gamma,x) = (1+\tilde{\kappa}(\gamma,x))\prod_k \frac{\sqrt{(\lambda-E_k)(\lambda-E_k^+)}}{(\lambda-\lambda_k(x))} \tag{18}$$

$$\Xi(\lambda,x) = \pm\left(1+\sum_{k>0}\frac{s_k(x)}{\lambda^k}\right)+\sum_{k>0}\frac{q_k(x)}{\lambda^k} \quad \text{as} \quad \gamma\to\pm\infty. \tag{19}$$

(2) $\tilde{\lambda}_k(x) = \lambda_k(x)-\delta_k$, $|k|>K_1$.
(3) $\tilde{\nu}_k(x) = \sqrt{(\lambda_k(x)-E_k)(\lambda_k(x)-E_k^+)}$, $|k|>K_1$.

These variables are dependent. Denote this set of variables by $\mathcal{S}$.

Consider the following norm:

$$\|\mathcal{S}\|_n = \sqrt{\sum_{|k|\leq K_1}(|s_k|^2+|q_k|^2)+\sum_{|k|>K_1}|k|^n\left(|\tilde{\lambda}_k|^2+|\tilde{\nu}_k|^2\right)} \tag{20}$$

This norm is bounded on the space of admissible divisors for any positive n.

Lemma 11 *For any sufficiently large n there exists a constant $C_n(\Gamma)$ such, that for any admissible pair $\mathcal{S}_1(x)$, $\mathcal{S}_2(x)$ of solutions of Dubrovin equations we have the following estimate*

$$\left\|\frac{\partial}{\partial x}\left(\mathcal{S}_1(x)-\mathcal{S}_2(x)\right)\right\|_n \leq C_n(\Gamma)\|\mathcal{S}_1(x)-\mathcal{S}_2(x)\|_n. \tag{21}$$

It is easy to check, that for any $\epsilon_1>0$ there exists a constant $K_3(n)$ such that

$$\|\mathcal{S}\|_n^{(2)}<\epsilon_1, \quad \text{where} \quad \|\mathcal{S}\|_n^{(2)} = \sqrt{\sum_{|k|>K_3(n)}|k|^n\left(|\tilde{\lambda}_k|^2+|\tilde{\nu}_k|^2\right)}. \tag{22}$$

We need also the following semi-norm

$$\|\mathcal{S}\|_n^{(1)} = \sqrt{\sum_{|k|\leq K_1}(|s_k|^2+|q_k|^2)+\sum_{K_1<|k|<K_3(n)}|k|^n\left(|\tilde{\lambda}_k|^2+|\tilde{\nu}_k|^2\right)}. \tag{23}$$

It is evident, that $\|\mathcal{S}\|_n^{\leq}\|\mathcal{S}\|_n^{(1)} + \|\mathcal{S}\|_n^{(2)}$ and $\|\mathcal{S}\|_n^{(1)} \leq \ \|\mathcal{S}\|_n$.

Combining (21) and (22) we obtain the following estimate

$$\|\mathcal{S}_1(x) - \mathcal{S}_2(x)\|_n \leq e^{C_n(\Gamma)|x|} \left(\|\mathcal{S}_1(0) - \mathcal{S}_2(0)\|_n^{(1)} + \epsilon_1\right). \tag{24}$$

Therefore in approximate calculations we can truncate the Dubrovin system to a finite-dimensional one, removing the variables $\tilde{\lambda}_k(x)$, $\tilde{\nu}_k(x)$, with the $|k| > K_3(n)$. For the truncated system small variations of the curve and of the starting point result in small variations of the solution. To complete the proof it is sufficient to check, that choosing K sufficiently large we can make the admissible variation of divisor arbitrary small. But it follows from the characterization of admissible divisors presented above.

These arguments can be applied also for the Dubrovin equations, describing the t_l -evolution of the divisor, where t_l denotes the l-s time from the NLS hierarchy (in these notations $t = t_1$). Taking into account that the first k x-derivatives of $q(x)$ are continuous functionals in the norm $\| \ \|_n$ for sufficiently large n we obtain:

Theorem 1 *Let $q(x,t)$ be an arbitrary SfNLS solution with smooth x-periodic Cauchy data $q(x,0) = q_0(x)$. Then for any $\epsilon > 0$, $N > 0$ and $\mathcal{T} > 0$ there exists a finite-gap SfNLS solution $q^F(x,t)$ such, that*

$$\left| \frac{\partial^n}{\partial x^n} \left(q^F(x,t) - q(x,t)\right) \right| < \epsilon \quad \textit{for all} \quad x \in \mathbb{R},\ |t| < \mathcal{T},\ n \leq N. \tag{25}$$

At the end let us say a few words how to prove an analogous theorem for the Filament equation. Equivalence between the SfNLS and the Filament equation is given by the Hasimoto map [5]. (θ-functional solutions of Filament equations were studied by Sym in [13]).

$$q(s,t) = \frac{1}{2} k(s,t) e^{i\int^s \kappa(\tilde{s},t)d\tilde{s}}, \tag{26}$$

where $k(s,t)$, $\kappa(s,t)$ are the curvature and the torsion functions respectively. In [4] it was shown, that equations (17) (except one corresponding to $\alpha_0 = 0$) preserve the periodicity in s of the Filament equation solution. Therefore the technique developed above can be applied without changes.

Acknowledgments. The author is grateful to Martin Schmidt for explaining some results from the periodic NLS theory.

References

[1] I. V. Cerednik. On the conditions of reality in "finite-gap integration". (Russian) *Dokl. Akad. Nauk SSSR*, **252, no. 5** (1980), 1104–1108.

[2] L. D. Faddeev, L. A. Takhtajian. *Hamiltonian methods in the theory of solitons*, Springer-Verlag, 1980.

[3] P. G. Grinevich, M. U. Schmidt. Period preserving nonisospectral flows and the moduli space of periodic solutions of soliton equations. *Physica D,* **87** (1995), 73-98.

[4] Grinevich P.G., Schmidt M.U. Closed curves in R^3: a characterization in terms of curvature and torsion, the Hasimoto map and periodic solutions of the Filament Equation. — SFB 288 preprint No 254; Electronic version: dg-ga@msri.org/9703020.

[5] R. Hasimoto. A soliton on vortex filament, J. Fluid. Mechanics, **51** (1972), 477-485.

[6] A. R. Its., V. P. Kotljarov. Explicit formulas for solutions of the Nonlinear Schrödinger equation. (Ukrainian) — *Dokl. Ukrain. SSR, Ser. A*, **no. 11**, (1976), 965–968.

[7] A. M. Kamchatnov, On improving the effectiveness of periodic solutions of the NLS and DNLS equations. *J. Phys. A*, **23, no. 13** (1990), 2945–2960.

[8] I. M. Krichever. The Cauchy problem for doubly periodic solutions of KP II equation. Important developments in soliton theory, ed. A.Fokas and V.E.Zakharov, Springer-Verlag, 1993, 123-146.

[9] V. A. Marchenko, I. V. Ostrovski. A characterization of the spectrum of Hill's operator. *Math. USSR Sb.* **26:4** (1975) 493-554.

[10] S. P. Novikov. The periodic Korteveg-de Vries problem. I. *Funct. Anal. Appl.* **8** (1974), 236-246.

[11] E. Previato. Hyperelliptic quasi-periodic and soliton solutions of the nonlinear Schrödinger equation, *Duce Math. Journal,* **52:2** (1985), 329-377.

[12] M. U. Schmidt. Integrable systems and Riemann surfaces of infinite genus. *Memoirs of the American Mathematical Society,* **551** (1996).

[13] Sym A. Vortex filament motion in terms of Jacobi theta functions. — *Fluid Dynamics research*, **3** (1988), 151-156.

[14] V. E. Zakharov, S. V. Manakov, S. P. Novikov, L. P. Pitaevsky. *Soliton Theory,* New York: Plenum 1984.

[15] V.E.Zakharov, A. B. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Soviet Phys. Jetp,* **34** (1972), 62-69.